Determination of the factors causing the development of aseptic instability of endoprosthesis components is one of the main tasks of modern traumatology and orthopedics. It is important from a scientific and medical point of view to carry out physical and mathematical modeling of the distribution of load forces and their moments on the femoral component of a hip endoprosthesis. The purpose of the study: to conduct a physical and mathematical modeling of the distribution of load forces on the femoral component of a hip endoprosthesis under real conditions of incomplete axially symmetrical contact of the femoral component of the endoprosthesis and the femur, when the surface of the lower end of the endoprosthesis is not in contact with the surface of the bone. In the work, mathematical modeling of the distribution of point load forces and their moments on the contact surface between the femoral endoprosthesis stem and the bone marrow canal of the femur in real conditions is carried out. For qualitative estimates of point distributions of the load force, based on the analysis of previous results, an estimated empirical formula was obtained for these distributions: \( g(\lambda, \lambda_1) = \frac{P(\lambda, \lambda_1)}{140\lambda^{7/2}} \), in which the pressure \( P(\lambda, \lambda_1) \) is taken in kilograms per square centimeter (kg/cm²), and the point force \( g(\lambda, \lambda_1) \) is in kilograms (kg). It was determined that the best, from the point of view of minimizing the harmful mechanical impact of the prosthesis on the femur, is the situation when the length of the prosthesis stem is not less than half the length of the femur (\( \lambda > 0.5 \)). In this case, the values of the point load forces do not exceed 0.1 kg, at least for the length of the area of real contact, which is not less than half the length of the prosthesis stem (\( \lambda_1 > 0.5\lambda \)). It has been proven that the use of a prosthesis stem that is less than a third of the length of the femur is not advisable. Since already at the length of the prosthesis stem, which is 30 % of the length of the femur (\( \lambda = 0.3 \)), point loads increase rapidly and can reach from 0.55 to 1.5 kg depending on the length of the contact area. Such point loads are undesirable for the femur in the area of contact with the prosthesis in terms of the integrity of the femur.

**Keywords:** bone tissue, femoral implant, point distribution of load force, mathematical model in real conditions.
optimal indicators of the load on the implanted femoral component of the endoprosthesis. This leads in the early postoperative period during rehabilitation loads to underloading or overloading of the bone-femoral component of the endoprosthesis system with impaired osseointegration of bone tissue.

The adaptive properties of bone tissue, depending on its external load, are a powerful means for the body to restore impaired functions of the bone system [3, 5]. However, in diagnostic studies used today, these properties are taken into account subjectively. The reason for this provision is the complexity of the adaptation process and the lack of technical means of control over the adaptive change in the structure and mechanical characteristics of living bone. Currently, the only means of predicting the reaction of bone tissue to changes in external mechanical load is biomechanical modeling [8, 20]. The method of mathematical modeling makes it possible to eliminate the need for the production of bulky physical models associated with material costs; to reduce the time of determining the characteristics (especially when calculating mathematical models using computer technologies and effective computational methods and algorithms); study the behavior of the modeling object at different parameter values, predicting the nature of its changes from the analysis of the mathematical model; analyze the possibility of using different elements; obtain characteristics and indicators that are difficult to obtain experimentally (correlation, frequency, parametric sensitivity) [15]. Therefore, it is important from a scientific and medical point of view to carry out physical-mathematical modeling of the distribution of load forces and their moments on the femoral component of the endoprosthesis of the hip joint.

The main attention in the work is focused on the calculations and analysis of the dependence of the distribution of elastic load forces on the length of the endoprosthesis stem in real conditions of incomplete contact of the femoral component of the endoprosthesis and the femur and the absence of defects in the proximal part of the femur. The distribution of forces corresponding to such a real contact can occur immediately after the operation and be a quantitative measure of the quality of its execution. Deviation from such distributions is also a quantitative measure of the quality of its execution.

The purpose of the research is to carry out physical and mathematical modeling of the distribution of load forces on the femoral component of the hip joint under real conditions of incomplete axially symmetrical contact of the femoral component of the endoprosthesis and the femur, when the surface of the lower end of the endoprosthesis is not in contact with the surface of the bone.

Materials and methods

The article was carried out with the funds of the state budget of Ukraine, the research was carried out within the framework of the State Institution "Institute of Traumatology and Orthopedics of the National Academy of Medical Sciences of Ukraine" "Develop new and improve existing methods of diagnosis and treatment of patients with coxarthrosis with accompanying spinal pathology" (2019-2021) Subject code CF.2020.1.NAMSU (applicable), state registration № 0119U001022.

The experimental research plan was approved by the Bioethics Committee of the State Institution "Institute of Traumatology and Orthopedics of the National Academy of Medical Sciences of Ukraine" (Protocol № 198 dated 10.05.2020).

Description of the model and preliminary calculations

1. Modification of the model

The femur is elongated along the z axis and has a length $L_0=40$ cm (Fig. 1). The radius of the cylindrical medullary canal of the femur is assumed to be equal to $R_0=1$ cm. Figure 1 shows the idealized situation for the length of the stem prosthesis $\Lambda=20$ cm length, but here too this value is used as one of the main variable parameters, the influence of which on the distribution of forces is studied [11].

The conical contact surface is modeled by a hyperboloid of rotation:

$$z(r) = \frac{L_0}{R_0} \sqrt{r^2 + \frac{R_0^2}{L_0^2}(L_0 - \Lambda)^2}$$

(1)

bounded from above by a horizontal plane: $z(r)=L_0$

In a real situation, it is taken into account that the lower part of the stem prosthesis is not in contact with the inner surface of the femur. Only the upper part of the prosthesis stem is in contact $\Lambda_1$ long. It is obvious that the inequality always holds $\Lambda_1 \leq \Lambda$. That is, only part of the surface (1) that satisfies the condition is considered $L_0 \leq z(r) \leq L_0$

Figure 2 shows the same situation as Figure 1, but for real contact area. Value $\Lambda_1$ chosen equal to $15$ cm. That is, in relation to the value selected in Figure 1 $\Lambda=20$ cm, Figure 2 shows a case $\Lambda_1=0.75$

2. Determination of the real contact area of the prosthesis stem and the medullary canal of the femur

Now the task is to find the area of the surface shown in Figure 2 and bounded from below and above by horizontal planes $z(r)=L_0/\Lambda$ and $z(r)=L_0$ respectively. That is, the surface (1) we are considering is limited by inequalities:

$$(L_0 - \Lambda_1) \leq z(r) \leq L_0$$

(2)

As before [14], we first find an infinitesimally small element of the surface (1). In accordance with the definition of G. Korn and T. Korn. [9] its projection on the plane $(x,y)$ (or $(r,\phi)$ in cylindrical coordinates) is determined by the ratio:

$$dS = \sqrt{1 + (\frac{\partial z}{\partial r})^2 + (\frac{\partial z}{\partial \phi})^2} \, dr \, d\phi$$

(3)

where $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \phi}$ - partial derivatives of the surface equation (1), which here must first be used in the form:

$$z(x,y) = \frac{L_0}{R_0} \sqrt{x^2 + y^2 + \frac{R_0^2}{L_0^2}(L_0 - \Lambda)^2}$$
To find the area \( S \) of the surface contact, it is most convenient to use the cylindrical representation:

\[
dS(r, \varphi) = \sqrt{1 + \frac{l_0^2 r^2}{r^2 + R_0^2 (1 - \lambda)^2}} r \, dr \, d\varphi.
\]  

(6)

In this representation of integration over the angular variable \( \varphi \) from 0 to \( 2\pi \) can be performed immediately, since nothing depends on it. That is, such an integral will give simply a factor \( 2\pi \). As for integration by variable \( r \), then here the integration is performed within the range of values that now depends on two parameters. From the dimensionless length of the prosthesis stem \( \lambda \) and the length of the contact area \( \lambda_1 = \lambda l / l_0 \) and are determined by condition (2). To find the limits of integration, we will substitute (1) in (2) and solve the obtained relation \( r \) for both parts of inequality (2). As a result, we will have a value \( R_1 \) and \( R_{1,\text{ur}} \), respectively, for the upper and lower limits of integration:

\[
R_1 = \sqrt{1 - (1 - \lambda)^2}; \quad R_{1,\text{ur}} = \sqrt{1 - (1 - \lambda)^2} - (1 - \lambda),
\]

(7)

and the integral itself for determining the required area is reduced to the form:

\[
S(\lambda, \lambda_1) = \pi R_1 \int_{\lambda}^{\lambda_1} \left[ \frac{1}{(1 - \lambda)^2} \right] r \, dr.
\]

(8)

In the last entry, it is emphasized that the dependence of the area \( S \) is investigated not only on the dimensionless length of the prosthesis stem: \( \lambda = \Lambda / l_0 \), and from the dimensionless length of the contact region: \( \lambda_1 = \lambda l / l_0 \). Figure 3 shows the dependence \( S(\lambda, \lambda_1) \) calculated according to formula (8). The highest curve \( (\lambda_1 = \lambda) \) in Figure 3 corresponds to the ideal situation of full surface contact [11]. From the graphs in Figure 3, it can be seen that when the area of real contact is not less than 75 % of the length of the prosthesis stem \( (\lambda_1 < 0.75\lambda) \), then the value of the area differs little from the ideal situation. The contact area decreases significantly when the length of the contact area decreases \( \lambda_1 \) (in Figure 3 are curves \( \lambda_1 = 0.5\lambda \) and \( \lambda_1 = 0.25\lambda \)).
Results

Determination of pressure and point distribution of load forces in the “prosthesis-femur” system under real conditions and analysis of its influence on mechanical stability.

Now, knowing the contact area (8), it is necessary to find the basic characteristics - the pressure on the contact surface of the stem prosthesis and the femur and point distributions of load forces.

Pressure is defined as the ratio of the magnitude of the load force \( F_0 \) to the contact area \( S(\lambda, \lambda_1) \):

\[
P(\lambda, \lambda_1) = \frac{F_0}{S(\lambda, \lambda_1)},
\]

where the area is defined by the integral (8). As before, the basic load force is assumed to be equal: \( F_0 = 40 \text{ kg} \).

Dependence of pressure on dimensionless length \( \lambda \) of the prosthesis stem for different values of the contact area \( \lambda_1 \) shown in Figure 4.

Here, the lowest curve 1 corresponds to the ideal case \([11] (\lambda_1 = \lambda)\). Curve 2, which corresponds to the contact area, which is 75% of the length of the prosthesis stem \((\lambda_1 = 0.75 \lambda)\) is quite closely adjacent to it. As can be seen from the figure, the area: \(0.75 \lambda < \lambda_1 < \lambda\) (when the length of the contact surface exceeds 75% of the total length of the stem of the prosthesis) is the best, as it provides the least pressure on the femur from the side of the prosthesis. Further, with the reduction of the contact area, the pressure increases quite quickly, which is already undesirable from the point of view of the integrity of the femur.

For qualitative estimates of point distributions of the load force, based on the analysis of previous results of T. V. Nizalov and co-authors [11], an estimated empirical formula was obtained for these distributions:

\[
g(\lambda, \lambda_1) = \frac{P(\lambda, \lambda_1)}{140 \lambda^{1/2}}
\]

in which the pressure \( P(\lambda, \lambda_1) \) is taken in kilograms per square centimeter \((\text{kg/cm}^2)\), and point force \( g(\lambda, \lambda_1) \) - in kilograms \((\text{kg})\). The distribution of this force is shown in Figure 5.

Figure 5 shows that the best situation, from the point of view of minimizing the harmful mechanical impact of the prosthesis on the femur, is when the length of the stem of the prosthesis is not less than half the length of the femur \((\lambda > 0.5)\). In this case, the values of point load forces do not exceed 0.1 kg, at least for the length of the area of real contact, which is not less than half the length of the stem \((\lambda > 0.5\lambda_1)\). As can be seen from Figure 5 (curve 4), already for the length of the contact area, which is a quarter of the length of the prosthesis stem \((\lambda_1 = 0.25 \lambda)\) the point force can reach 0.15 kg for the stem length of the prosthesis \((\lambda = 0.5)\). From the graphs in Figure 5, it is also clear that the length of the prosthesis stem, which is less than one third of the length of the femur, is undesirable. Because already at the length of the prosthesis stem, which is 30% of the length of the femur \((\lambda = 0.3)\), point loads grow rapidly and, as can be seen from the graphs in Figure 5, can reach from 0.55 to 1.5 kg, depending on the length of the contact area. Such point loads are undesirable for the femur in the area of contact with the prosthesis in terms of the integrity of the femur.

Discussion

Mathematical models of the theory of reliability are models that take into account mechanical, physical and other real processes that entail a change in the properties of the object and its constituent parts. These are mechanics...
models that are widely used in structural calculations. Force and kinematic interactions of elements and structures are complex. The behavior of these objects significantly depends on their interaction with the environment, the nature and intensity of operational processes [4].

To predict the behavior of structural elements, it is necessary to take into account the processes of loading, deformation, wear, accumulation of damage and destruction under variable loads and other external influences. It is possible to evaluate system reliability indicators theoretically and computationally based on physical models and statistical data on material properties, loads and impacts [1].

On the basis of previously developed methods of mathematical modeling of load forces for stem endoprosthesis of the hip joint under conditions of ideal contact, similar calculations were carried out under conditions of real contact for pressure on the femur from the side of the stem endoprosthesis and point distributions of load forces. Namely, under conditions when the area of the top of the femoral component of the endoprosthesis is not in contact with the femur.

The area of real contact between the stem prosthesis and the medullary canal of the femur, when the contact surface is not completely dense in the area of the distal end of the stem prosthesis, was modeled in the form of a truncated hyperboloid of rotation, which is the closest geometric shape of this part of the prosthesis in real conditions. The truncation plane was considered perpendicular to the axis of the femoral component of the endoprosthesis. The main attention was focused on the analysis of the dependence of the real contact area, the pressure on the femur from the endoprosthesis, and the point distributions of load forces on the length of the stem prosthesis and the contact area. Deviations in the distribution of pressure and point load forces under conditions of real contact due to changes in the contact area caused by disturbances in different zones in accordance with the classification of W. G. Paproski and co-authors [13] is a quantitative measure of pathological changes during the operation of the endoprosthesis.

Comparison of the observed changes in the contact area with the results of pressure calculations and point distributions of load forces for such changes may in the future provide an opportunity to develop a diagnostic and treatment algorithm and determine the need for additional surgical intervention.

The conducted research will contribute to the development of a diagnostic and treatment algorithm, the implementation of which will increase the effectiveness of the treatment of this severe orthopedic pathology.

**Conclusion**

1. In the work, mathematical modeling of the distribution of point load forces and their moments on the contact surface between the femoral endoprosthesis stem and the medullary canal of the femur under real conditions is carried out.

2. It was determined that the best situation, from the point of view of minimizing the harmful mechanical impact of the prosthesis on the femur, is the situation when the length of the stem of the prosthesis is not less than half the length of the femur ($\lambda \geq 0.5$). In this case, the values of point load forces do not exceed 0.1 kg, at least for the length of the area of real contact, which is not less than half the length of the prosthesis stem ($\lambda \geq 0.5\lambda$).

3. It has been proven that the use of a stem prosthesis that is less than a third of the length of the femur is not advisable. Because already at the length of the prosthesis stem, which is 30 % of the length of the femur ($\lambda = 0.3$), point loads grow rapidly and can reach from 0.55 to 1.5 kg, depending on the length of the contact area. Such point loads are undesirable for the femur in the area of contact with the prosthesis in terms of the integrity of the femur.

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ФІЗИКО-МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ РОЗПОДІЛУ СИЛ НАВАНТЖЕННЯ НА СТЕГНОВОМУ КОМПОНЕНТІ ЕНДОПРОТЕЗА КУЛЬШОВОГО СУГЛОБА ЗА РЕАЛЬНИХ УМОВ

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Визначення факторів, що зумовлюють розвиток онкологічної нестабільності компонентів ендопротезу є одним з основних завдань сучасної травматології та ортопедії. Важливим в науковому та медичному плані є проведення фізико-математичного моделювання розподілу сил навантаження на стегновому компоненті ендопротеза кульшового суглоба. Метою дослідження є вивчення впливу протезу на стегнову кістку, коли поверхня нижнього кінця ендопротезу знаходиться не в контакті з поверхнею кістки. У роботі проведено математичне моделювання розподілу точкових сил навантаження та їх моментів на поверхні контакту між нижкою ендопротезу кульшового суглоба та кістково-мозковим каналом стегнової кістки в реальних умовах.

Ключові слова: стегнова тканина, стегновий імплант, точковий розподіл сил навантаження, математична модель в реальних умовах.

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